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A METHOD OF MAKING IMPEDANCE MEASUREMENTS OF THE VISCOELASTIC PROPERTIES OF A MEDIUM BY OSCILLATING PLATES AND SHELLS*

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A method of measuring the viscoelastic properties of a homogeneous medium bounded by plates and shells is presented, based on processing observations of arbitrary oscillations.

The impedance measurements an attempt is usually made to use the simplest forms of oscillations and obtain one-dimensional motion. However if the region in which oscillations are excited is of limited size this gives rise to difficulties due to diffraction or edge effects, and the excitation of modes of oscillation that are not used in the measurements. An increase in the dimensions of the excitation region in order to reduce the influence of these effects usually requires an increase in the stiffness, and hence also in the mass of the probes, which reduces the measurement sensitivity, and makes them virtually impossible at high frequencies.

A method is propsoed in the present paper of processing the observed arbitrary oscillations of plates and shells, which enable us to obtain the same results and formulas for the simplest modes of oscillation including one-dimensional modes. This is also feasible in cases for which this realization is practically impossible, which enables the range of impedance frequency measurements to be extended. The viscoelastic properties of the medium are determined in terms of displacements and stresses on the plate or shell surfaces for arbitrarily small oscillations.

Let the oscillations of a homogeneous plate or shell with bounding surfaces S_1 and S_2 be used for impedance measurements, where the surface S_2 is in contact with the viscoelastic medium being investigated. The oscillations observed on the surface S_1 and their properties are used to determine the properties of the medium. Such plates or shells can be, for example, the walls of apparatus, autoclaves, containers, or pipelines, the values of underground structures, or natural objects. We shall restrict our consideration to the simplest configurations of plates and shells.

The equations of small oscillations of a plate (shell) and medium have the form /1/

$$k_j^{-2}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}_j - \varkappa_j^{-2}\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u}_j + \mathbf{u}_j = 0 \tag{1}$$

$$k_j = \omega/c_j,\quad \varkappa_j = \omega/v_j,\quad c_j = \sqrt{(\lambda_j + 2\mu_j)/\rho_j},\quad v_j = \sqrt{\mu_j/\rho_j}$$

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When $j=1$, the equation is considered in the region occupied by the plate or shell, and when $j=2$ in the region occupied by the medium, $\mathbf{u}_j$ are the complex amplitudes or spectral densities of the displacement vectors (subsequently called displacements), k_j and $\varkappa_j$ are the wave numbers for plane longitudinal and transverse waves, respectively, that propagate at complex velocities c_j and v_j dependent on the frequency, λ_j and μ_j are complex Lamé coefficients, dependent on frequency, ρ_j are the densities of the media, ω is the angular frequency, and the multiplier $\exp(-i\omega t)$ is everywhere omitted.

The vector of stresses on a certain surface S with unit normal $\mathbf{n}$ is expressed in terms of the displacement vectors $\mathbf{u}_j$ by the formula /2/

$$\sigma^{(j)}=2\mu_j\frac{\partial\mathbf{u}^{(j)}}{\partial n}+\lambda_j\mathbf{n}\operatorname{div}\mathbf{u}^{(j)}+\mu_j[\mathbf{n}\times\operatorname{rot}\mathbf{u}^{(j)}] \tag{2}$$

On the interface S_2 of the plate or shell and the medium all components of the displacement and stress vectors are identical

$$\mathbf{u}^{(1)}(S_2)=\mathbf{u}^{(2)}(S_2),\quad \sigma^{(1)}(S_2)=\sigma^{(2)}(S_2) \tag{3}$$

Since λ_j and μ_j are not real, then, when the medium is unbounded, the conditions of damping at infinity

$$\sqrt{x_1^2+x_2^2+x_3^2}\to\infty,\quad |\mathbf{u}^{(j)}|\to 0 \tag{4}$$

are assumed to be satisfied.

If some of the constants λ_j and μ_j are taken to be real, the damping condition is replaced by the principle of limit absorption /3, 4/.

We will illustrate the basic idea of this paper using the simplest example of the determination of the characteristics of a homogeneous viscoelastic half-space $(z>0)$ in contact with a plane viscoplastic layer $(-h\leqslant z\leqslant 0)$.

We shall further need three simple exact solutions of problem (1)–(3) using which we shall illustrate the essential features of the proposed system of measurements. These solutions will be called the reference oscillations.

Normal uniform oscillations (independent of x and y) of the medium and layer are defined by the formulas

$$\begin{gathered}\mathbf{u}^{(j)}=\{0,0,w^{(j)}\}\\ w^{(1)}=A\left(\cos k_1z+i\frac{\rho_2c_2}{\rho_1c_1}\sin k_1z\right),\quad w^{(2)}=Ae^{ik_2z};\quad A=A(\omega)\end{gathered} \tag{5}$$

From this, using (2), we obtain

$$\sigma^{(1)}|_{z=-h}=(0,0,\sigma),\quad \sigma=\omega A\,(\rho_1c_1\sin k_1h+i\rho_2c_2\cos k_1h) \tag{6}$$

For tangential uniform oscillations of the medium and layer along the x axis we have

$$\mathbf{u}^{(j)}=\{u^{(j)},0,0\} \tag{7}$$

$$u^{(1)}=A\left(\cos\varkappa_1z+i\frac{\rho_2v_2}{\rho_1v_1}\sin\varkappa_1z\right),\quad u^{(2)}=Ae^{i\varkappa_2z};\quad A=A(\omega)$$

$$\sigma^{(1)}|_{z=-h}=(\tau,0,0),\quad \tau=\omega A\,(\rho_1v_1\sin\varkappa_1h+i\rho_2v_2\cos\varkappa_1h) \tag{8}$$

For uniform torsional oscillations (the angle of rotation $\varphi^{(j)}$ is independent of x and y) of the medium and layer about the z axis we have

$$\mathbf{u}^{(j)}=(-y,x,0)\,\varphi^{(j)} \tag{9}$$

$$\varphi^{(1)}=A\left(\cos\varkappa_1z+i\frac{\rho_2v_2}{\rho_1v_1}\sin\varkappa_1z\right),\quad \varphi^{(2)}=Ae^{i\varkappa_2z};\quad A=A(\omega)$$

$$\sigma^{(1)}|_{z=-h}=(y,-x,0)\,M,\quad M=\omega A\,(\rho_1v_1\sin\varkappa_1h+i\rho_2v_2\times\sin\varkappa_1h) \tag{10}$$

If these simple motions can be realised, by measuring on the surface of observation $z=-h$ the specific impedance

$$z_n=\frac{\sigma^{(1)}}{i\omega\mathbf{u}^{(1)}}\bigg|_{z=-h}=-\frac{\sigma^{(1)}}{\mathbf{v}^{(1)}}\bigg|_{z=-h} \tag{11}$$

where $\mathbf{v}^{(1)}$ is the velocity vector, it should be possible, for known characteristics of the plate, to determine the properties of the medium.

Let z_1, z_2 and z_3 be the specific impedances of the three reference oscillations considered above. Then from formulas (5)–(11) we obtain

$$\rho_2 c_2 = \rho_1 c_1 \frac{z_1 + i\rho_1 c_1 \operatorname{tg} k_1 h}{\rho_1 c_1 + i z_1 \operatorname{tg} k_1 h} \quad (n = 1) \tag{12}$$

$$\rho_2 v_2 = \rho_1 v_1 \frac{z_n + i\rho_1 v_1 \operatorname{tg} \varkappa_1 h}{\rho_1 v_1 + i z_n \operatorname{tg} \varkappa_1 h} \quad (n = 2, 3) \tag{13}$$

It is not possible to realize the simple forms of socillations considered, but there is no need for this. Indeed, let $(\mathbf{U}^{(1)}, \mathbf{U}^{(2)})$ be an arbitrary, motion as complicated as desired of the system, and suppose that on the observation surface its displacement $\mathbf{U}$ (or velocity) and stress Σ are recorded. Since $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ satisfy (1) as well as the boundary conditions (3) and (4), it follows from Green's formula for the solution $(\mathbf{U}^{(1)}, \mathbf{U}^{(2)})$ and solutions (5) and (7), or (9) that

$$\iint\limits_{z=-h} (\mathbf{u}^{(1)}, \Sigma)\, dx\, dy = \iint\limits_{z=-h} (\mathbf{U}, \sigma^{(1)})\, dx\, dy \tag{14}$$

where $(\mathbf{a}, \mathbf{b})$ is the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Substituting here $\mathbf{u}^{(1)}$ and $\sigma^{(1)}$ from (5) – (10) and solving the formulas obtained for $\rho_2 c_2$ and $\rho_2 v_2$, we obtain (12) and (13) in which z_n is replaced by the following quantities Z_n:

$$Z_n = \frac{I_n(\Sigma)}{i\omega I_n(\mathbf{U})} = -\frac{I_n(\Sigma)}{I_n(\mathbf{V})}, \quad I_n(\mathbf{B}) = \iint\limits_{z=-h} (\mathbf{B}, \alpha_n)\, dx\, dy$$

$$\mathbf{V} = -i\omega\mathbf{U}, \ \alpha_1 = (0, 0, 1), \ \alpha_2 = (1, 0, 0), \ \alpha_3 = (-y, x, 0)$$

and it is assumed here that the denominator does not vanish.

Hence using (12) and (13) with Z_n substituted for z_n it is possible to determine the medium parameters from the results of arbitrary oscillations of the plate. Taking into account the natural losses in the elastic plates and when the medium possesses viscosity, the results obtained hold without any restrictions. If, however, any of the quantities k_j or $\varkappa_j$ are taken as real, for these results to be valid it is necessary that the observed oscillations should fall fairly rapidly when $\sqrt{x^2 + y^2 + z^2} \to \infty$. It is convenient in particular, to use non-uniform waves that are cylindrically divergent in the plate.

The case of a thin plate, when $|\varkappa_1 h| \ll 1$ is of considerable interest. From (12) and (13) we have

$$\rho_2 c_2 \approx \rho_1 c_1 \frac{ik_1 h + Z_1/(\rho_1 c_1)}{1 + ik_1 h Z_1/(\rho_1 c_1)}, \quad \rho_2 v_2 \approx \frac{i\varkappa_1 h + Z_{2,3}/(\rho_1 v_1)}{1 + i\varkappa_1 h Z_{2,3}/(\rho_1 v_1)}$$

In the zeroth approximation we have

$$\rho_2 c_2 \approx Z_1, \ \rho_2 v_2 \approx Z_{2,3} \tag{15}$$

Formulas (15) holds when investigating the viscoelastic properties of the medium in a half-space without using a plate.

Let us extend this principle to more complicated configurations of shells. Let $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ be some fairly simple (reference) solution of problem (1), (3), (4), and let $(\sigma^{(1)}, \sigma^{(2)})$ be the corresponding stress; $\mathbf{V}$ and Σ are the velocity and stress on the observation surface S_1 for arbitrary oscillations of the system. From Green's formula we have

$$-i\omega \iint\limits_{(S_1)} (\mathbf{u}^{(1)}, \Sigma)\, dS_1 = \iint\limits_{(S_1)} (\mathbf{V}, \sigma^{(1)})\, dS_1 \tag{16}$$

The stress and velocity on the surface S_1 are connected by the relation $\sigma^{(1)} = -\mathbf{z}\mathbf{v}^{(1)}$, where $\mathbf{z}$ is the impedance tensor for given reference oscillations. If the reference oscillations $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ are determined, it is possible to calculate the dependence of $\mathbf{z}$ on the medium properties, when the shell characteristics are known.

In the special case when the reference oscillations are such that $\mathbf{z} = z$, where $z = z(\rho_2, c_2, v_2)$ is a function independent of points on the surface S_1, we can take the function z outside the integral sign on the right-hand side of (16). Hence we have

$$z(\rho_2, c_2, v_2) = Z, \ Z = -\iint\limits_{(S_1)} (\Sigma, \mathbf{v}^{(1)})\, dS_1 \Big/ \iint\limits_{(S_1)} (\mathbf{V}, \mathbf{v}^{(1)})\, dS_1 \tag{17}$$

Thus, having determined the simplest solution $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ we find for it the dependence of the impedance on the medium parameters $z = z(\rho_2, c_2, v_2)$. Then having processed the results of observations of arbitrary oscillations on the surface S_1 (for which the denominator in the second formula of (17) is non-zero and this condition is assumed satisfied everywhere below) by the second formula (17), we determine the properties of the medium from the first of equations (17).

Let us use the principles described above to find the properties of a homogeneous viscoelastic medium contained in a cylindrical pipe. In cylindrical coordinates r, θ, z the displacement and stress vectors are represented in the form $\mathbf{u}^{(j)} = \{u_r^{(j)}, u_\theta^{(j)}, u_z^{(j)}\}$, $\sigma_r^{(j)} = \{\sigma^{(j)}, \sigma_\theta^{(j)}, \sigma_z^{(j)}\}$.

Using as reference oscillations the axisymmetric radial oscillations of the shell $(a \leqslant r \leqslant R)$ and medium $(r \leqslant a)$ in the form $\mathbf{u}^{(j)} = \{u_r^{(j)}, 0, 0\}$, we obtain the formula

$$\frac{\rho_2 c_2 J_0(a_2)}{\rho_1 c_1 J_1(a_2)} + 2\left(\frac{\alpha_1}{b_1} - \frac{\rho_2 v_2}{b_2 \rho_1 c_1}\right) = \begin{vmatrix} J_0(a_1) & N_0(a_1) & 0 \\ J_1(R_1) & N_1(R_1) & 1 \\ J_0(R_1) & N_0(R_1) & \zeta \end{vmatrix} \cdot \begin{vmatrix} J_1(a_1) & N_1(a_1) & 0 \\ J_1(R_1) & N_1(R_1) & 1 \\ J_0(R_1) & N_0(R_1) & \zeta \end{vmatrix}^{-1} \tag{18}$$

$$a_j = k_j a, \quad R_1 = k_1 R, \quad \alpha_j = v_j / c_j, \quad b_j = \varkappa_j a, \quad B_1 = \varkappa_1 R$$

$$\zeta = \frac{iZ}{\rho_1 c_1} + \frac{2\alpha_1}{B_1}$$

where $(J_n(x)$ and $N_n(x)$ are Bessel and Neumann functions of order n. Here, according to (17)

$$Z = -\iint\limits_{(r=R)} \Sigma_r(R)\, d\theta\, dz \Big/ \iint\limits_{(r=R)} V_r(R)\, d\theta\, dz$$

where V_r and Σ_r are the radial velocity and stress components on the observation surface $r = R$ for arbitrary oscillations of the shell containing the medium.

If axial axisymmetric oscillations have been selected as the shell and medium reference oscillations of the form $\mathbf{u}^{(j)} = \{0, 0, u_z^{(j)}(r)\}$, we obtain for the viscoelastic properties of the medium the formula

$$\frac{\rho_2 v_2 J_1(b_2)}{\rho_1 v_1 J_0(b_2)} = \begin{vmatrix} J_1(b_1) & N_1(b_1) & 0 \\ J_0(B_1) & N_0(B_1) & -1 \\ J_1(B_1) & N_1(B_1) & \frac{iZ}{\rho_1 v_1} \end{vmatrix} \cdot \begin{vmatrix} J_0(b_1) & N_0(b_1) & 0 \\ J_0(B_1) & N_0(B_1) & -1 \\ J_1(B_1) & N_1(B_1) & \frac{iZ}{\rho_1 v_1} \end{vmatrix}^{-1} \tag{19}$$

$$Z = -\iint\limits_{(r=R)} \Sigma_z(R)\, d\theta\, dz \Big/ \iint\limits_{(r=R)} V_z(R)\, d\theta\, dz$$

For a thin shell $(|B_1 - b_1| \ll 1)$ this formula takes the form

$$\rho_2 v_2 J_1(b_2)/J_0(b_2) \approx -iZ$$

In many practical cases we have $|b_2| \gg 1$ and the specific shear impedance of the medium is

$$\rho_2 v_2 \approx Z \tag{20}$$

By selecting the torsional axisymmetric reference oscillations in the form $\mathbf{u}^{(j)} = \{0, u_\theta^{(j)}(r), 0\}$, we obtain a formula which is similar to (19), when the functions J_1, J_2, N_1, N_2 are substituted for J_0, J_1, N_0, N_1, and Σ_z, V_z for Σ_θ, V_θ, respectively. Hence for a thin shell we can write

$$\rho_2 v_2 J_2(b_2)/J_1(b_2) \approx -iZ$$

In the asymptotic case when $|b_2| \gg 1$ we obtain (20).

By selecting one-dimensional spherically symmetric radial reference oscillations $\mathbf{u}^{(j)} = \{u_r^{(j)}(r), 0, 0\}$ for a spherical viscoelastic shell $(a \leqslant r \leqslant R)$, investigated in the region $r \leqslant a$, we obtain,(in spherical coordinates r, θ, φ), the following restults:

$$\frac{\rho_2 c_2^2 \sin a_2}{\rho_1 c_1^2 j_1(a_2)} + 4\alpha_1^2\left(1 - \frac{\rho_2 v_2^2}{\rho_1 v_1^2}\right) = \begin{vmatrix} \sin a_1 & -\cos a_1 & 0 \\ j_1(R_1) & n_1(R_1) & 1 \\ \sin R_1 & -\cos R_1 & \xi \end{vmatrix} \cdot \begin{vmatrix} j_1(a_1) & n_1(a_1) & 0 \\ j_1(R_1) & n_1(R_1) & 1 \\ \sin R_1 & -\cos R_1 & \xi \end{vmatrix}^{-1}$$

$$Z = -\frac{\iint \Sigma_r(R) \sin\theta\, d\theta\, d\varphi}{\iint V_r(R) \sin\theta\, d\theta\, d\varphi}, \quad \xi = \frac{iZ}{\rho_1 c_1} R_1 + 4\alpha_1^2$$

where

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

are Bessel and Neumann spherical functions.

For torsional reference oscillations, symmetric about the diameter, of a system of form $\mathbf{u}^{(j)} = \{0, 0, u_\varphi^{(j)}(r) \sin\theta\}$, we have

$$\frac{\rho_2 v_2^2 \sin b_2}{\rho_1 v_1^2 j_1(b_2)} + 3\left(1 - \frac{\rho_2 v_2^2}{\rho_1 v_1^2}\right) = \begin{vmatrix} \sin b_1 & -\cos b_1 & 0 \\ j_1(B_1) & n_1(b_1) & 1 \\ \sin B_1 & -\cos B_1 & \eta \end{vmatrix} \cdot \begin{vmatrix} j_1(b_1) & n_1(b_1) & 0 \\ j_1(B_1) & n_1(B_1) & 1 \\ \sin B_1 & -\cos B_1 & \eta \end{vmatrix}^{-1}$$

$$Z = -\frac{\iint \Sigma_\varphi(R) \sin^2\theta\, d\theta\, d\varphi}{\iint V_\varphi(R) \sin^2\theta\, d\theta\, d\varphi}, \quad \eta = \frac{iZ}{\rho_1 v_1} B_1 + 3$$

More complex oscillations of systems may be used as references, for which it is possible to determine the viscoelastic properties and its density by a combination of calculation formulas.

Similarly, it is possible using (17), to obtain expressions for the medium characteristics, when the medium surrounds the shell, and the oscillations are recorded on its inner surface.

The proposed principle can also be extended to some other shell forms.

For the practical realization of the proposed method it is necessary to know the displacements and stresses on the observation surface S_1 and process the data obtained using the formulas proposed above.

Analog and discrete systems of three-dimensional processing have found wide application in acoustic measurements /5, 7/. An example of the use of a discrete system is the set of transducers of displacement (velocity, acceleration) on the surface of a technological apparatus shell (the acceptable pitch transducers is determined using Kotel'nikov's theorem). The displacement pickup (velocities, accelerations) and stresses (pickup elements of strain gauge) may alternate and a concurrent measurement of stresses and displacements does not cause any difficulties. Further data processing can be carried out on simple computing equipment.

Since the proposed method does not require the oscillations to be of any specific form, it is possible to excite the shell by a priori specified stresses (e.g., application of a point force) and measure only displacements.

In the analog form of the measurement system it is possible to use electromechanical transducers located around the shell and performing direct integration in analog form of shell displacements by the summation of emfs, currents, charges, magnetic fluxes, etc.

Note that since the form of the oscillations is arbitrary, it is possible to excite in the shell oscillations that decay rapidly with distance (non-uniform waves), while at the same time reducing the observation surface.

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REPRESENTATION IN TERMS OF p-ANALYTIC FUNCTIONS OF THE GENERAL SOLUTION OF EQUATIONS OF THE THEORY OF ELASTICITY OF A TRANSVERSELY ISOTROPIC BODY*

O.G. GOMAN

A general solution is given for the equations of the theory of elasticity in terms of p-analytic functions for a transversely isotropic body in a non-axisymmetric stress state. This representation was obtained in /1/ for an isotropic medium. For the transport medium a similar representation is known only for the axisymmetric problem /2—4/.

1. We shall call the function

$$f(z, r) = p(z, r) + iq(z, r) \equiv \binom{p}{q}_\alpha$$

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